Linear Algebra I

10/11/2023, Friday, 15:00 – 17:00

You are **NOT** allowed to use any type of calculators.

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1 Diagonalization $\quad 3+3+3+3+3+10 = 25$ pts

Consider the matrix

$$M = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}.$$

(a) Without computing eigenvalues of M, find the sum and product of them.

(b) Without computing eigenvalues of M, determine whether it is diagonalizable. (Justify your answer!)

(c) Find its characteristic polynomial.

(d) Show that -2 is an eigenvalue of M.

(e) Find all eigenvalues of M.

(f) Is M unitarily diagonalizable? If yes, find an orthogonal diagonalizer.

2 Subspaces of $\mathbb{R}^n$ $\quad 10 + (5+5) = 20$ pts

(a) Let $a, b \in \mathbb{R}$ and

$$S = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \mid (x_1 - ax_2)(x_1 - bx_2) = 0 \right\}.$$

Determine all values of a and b such that S is a subspace of $\mathbb{R}^2$.

(b) Let

$$S_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) \quad \text{and} \quad S_2 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right).$$

Find bases for the subspaces:

(i) $S_1 \cap S_2$

(ii) $S_1 + S_2$

3 Jordan canonical form

3 + 10 + 12 = 25 pts

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

(b) Is A diagonalizable? Why?

(c) Find the Jordan canonical form of A.

4 Eigenvalues/eigenvectors

20 pts

Let a, b, c, d be scalars and $M \in \mathbb{F}^{m\times m}$. Consider the matrices

$$K = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} aM & bM \\ cM & dM \end{bmatrix}.$$

Show that $\mu\lambda$ is an eigenvalue of N if μ is an eigenvalue of K and λ is an eigenvalue of M.

(HINT: Let $(\mu, \boldsymbol{x})$ be an eigenpair of K and $(\lambda, \boldsymbol{z})$ be an eigenpair of M. Construct an eigenvector of N corresponding to the eigenvalue $\mu\lambda$ in terms of $\boldsymbol{x}$ and $\boldsymbol{z}$.)

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10 pts free